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Math 12H HW 4.7: Half Angle Identities and Solving advanced trigonometric equations

Half Angle Identities:

$$\sin\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\tfrac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \tan\tfrac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Sum of Angle properties:

$$R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \qquad R\sin(\theta-\alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

1. Use the half angle identities to evaluate each of the following:

a) $\sin 15°$

$\sin\frac{30}{2} = \sqrt{\frac{1-\cos 30}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$

b) $\cos 22.5°$

$\cos\frac{45}{2} = \sqrt{\frac{1+\cos 45}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \boxed{\frac{\sqrt{2+\sqrt{2}}}{2}}$

c) $\sin 202.5°$

$\sin\frac{405}{2} = -\sqrt{\frac{1-\cos 405}{2}} = -\sqrt{\frac{1-\cos 45}{2}} = \boxed{\frac{-\sqrt{2-\sqrt{2}}}{2}}$

S A
T C

d) $\cos 202.5°$

S A
T C

$\cos\frac{405}{2} = -\sqrt{\frac{1+\cos 45}{2}} = \boxed{\frac{-\sqrt{2-\sqrt{2}}}{2}}$

e) $\sin 292.5°$

S A
T C

$\sin\frac{585}{2} = -\sqrt{\frac{1-\cos 225}{2}} = \boxed{\frac{-\sqrt{2+\sqrt{2}}}{2}}$

2. Solve for θ, where $0° \le \theta \le 360°$ by using the form $R\cos(\theta-\alpha)$ or $R\sin(\theta-\alpha)$

a) $3\cos\theta + 2\sin\theta = 1$

$R\cos(\theta-\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 3 \quad R\sin\alpha = 2$

$R^2 = 13$

$R = \sqrt{13}$

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 33.69°$

$=\sqrt{13}\cos(\theta - 33.69°) = 1$

$\theta - 33.69° = \cos^{-1}\left(\frac{1}{\sqrt{13}}\right) = 73.9°, 286.1°$

$\boxed{\theta = 107.6°, 319.8°}$

b) $\cos\theta + \sqrt{3}\sin\theta = 1$

$R\cos(\theta-\alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 1 \quad R\sin\alpha = \sqrt{3}$

$R^2 = 4$

$R = 2$

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = 60°$

$2\cos(\theta - 60°) = 1$

$\theta - 60° = \cos^{-1}\left(\frac{1}{2}\right) = 60°, 300°$

$\boxed{\theta_1 = 120°, 360°, 0°}$

Don't get θ and then find the othe answer.
First get both answers for θ-33.69° and THEN get θ. It make a difference in your answer.

c) $12\cos\theta - 5\sin\theta = 6.5$

$R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 12 \quad R\sin\alpha = 5$

$R^2 = 169$

$R = 13$

$\tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.62^\circ$

$13\cos(\theta + 22.62^\circ) = 6.5$

$\theta + 22.62^\circ = \cos^{-1}\left(\frac{6.5}{13}\right) = 60^\circ, 300^\circ$

$\boxed{\theta = 37.38^\circ, 277.38^\circ}$

d) $5\sin\theta - 3\cos\theta = 1$

$R\sin(\theta-\alpha) = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$R\cos\alpha = 5 \quad R\sin\alpha = 3$

$R^2 = 36$

$R = 6$

$\tan\alpha = \frac{3}{5} \Rightarrow \alpha = 30.96^\circ$

$6\sin(\theta - 30.96^\circ) = 1$

$\theta - 30.96^\circ = \sin^{-1}\left(\frac{1}{6}\right) = 9.59^\circ, 170.4^\circ$

$\boxed{\theta = 40.56^\circ, 201.37^\circ}$

e) $2\cos 2\theta + 3\sin 3\theta = 3$

$2(1 - 2\sin^2\theta) + 3(3\sin\theta - 4\sin^3\theta) = 3$

$-12\sin^3\theta - 4\sin^2\theta + 9\sin\theta - 1 = 0$

use graphing software to get sol.

f) $8\sin 2\theta - 6\sin 2\theta = 3$

$2\sin 2\theta = 3$

$\sin 2\theta = \frac{3}{2}$

$\emptyset$

3. Find the Maximum and Minimum of each expression for $0^\circ \le \theta \le 360^\circ$

a) $2\cos\theta + 3\sin\theta$

$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + \sin\theta\sin\alpha$

$R\cos\alpha = 2 \quad R\sin\alpha = 3$

$R^2 = 13$

$R = \sqrt{13}$

$= \sqrt{13}\cos(\theta - 56.31^\circ)$

$\tan\alpha = \frac{3}{2}$

$\alpha = 56.31^\circ$

$\boxed{\text{MAX}: (56.31^\circ, \sqrt{13})}$

$\boxed{\text{MIN}: (236.3^\circ, -\sqrt{13})}$

very silly mistake

b) $4\cos\theta - 3\sin\theta$

$R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 4 \quad R\sin\alpha = 3$

$R^2 = 25$

$R = 5$

$5\cos(\theta + 36.87^\circ)$

$\tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.87^\circ$

$\boxed{\begin{array}{l}\text{Max}: (323.1^\circ, 5) \\ \text{Min}: (143.1^\circ, -5)\end{array}}$

c) $\sqrt{3}\sin\theta - \cos\theta$

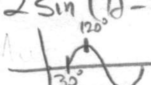
$R\sin(\theta-\alpha) = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$R\cos\alpha = \sqrt{3} \quad R\sin\alpha = 1$

$R^2 = 4$

$R = 2$

$2\sin(\theta - 30^\circ)$



$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = 30^\circ$

$\boxed{\begin{array}{l}\text{Max}: (120^\circ, 2) \\ \text{Min}: (300^\circ, -2)\end{array}}$

← max = 30+90

← min = 30+270

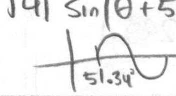
d) $4\sin\theta + 5\cos\theta$

$R\sin(\theta+\alpha) = R\sin\theta\cos\alpha + R\sin\alpha\cos\theta$

$R\cos\alpha = 4 \quad R\sin\alpha = 5$

$R^2 = 41 \Rightarrow R = \sqrt{41}$

$\sqrt{41}\sin(\theta + 51.34^\circ)$



$\tan\alpha = \frac{5}{4} \Rightarrow \alpha = 51.34^\circ$

$\boxed{\begin{array}{l}\text{Max}: (141.34^\circ, \sqrt{41}) \\ \text{Min}: (321.34^\circ, -\sqrt{41})\end{array}}$

e) $4+3\cos\theta-2\sin\theta$

$R\cos(\theta+\alpha)=R\cos\theta\cos\alpha-R\sin\theta\sin\alpha$

$R\cos\alpha=3 \quad R\sin\alpha=2$

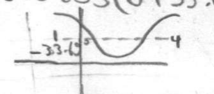
$R^2=13$

$R=\sqrt{13}$

$\sqrt{13}\cos(\theta+33.69^\circ)+4$

$\tan\alpha=\frac{2}{3}$

$\alpha=33.69^\circ$



$\boxed{\text{Max: }(326.3^\circ, 4+\sqrt{13}) \quad \text{Min: }(146.3^\circ, 4-\sqrt{13})}$ //

f) $2\cos\theta+\sin\theta-2$

$R\cos(\theta-\alpha)=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$

$R\cos\alpha=2 \quad R\sin\alpha=1$

$R^2=5$

$R=\sqrt{5}$

$\sqrt{5}\cos(\theta-26.57^\circ)-2$

$\tan\alpha=\frac{1}{2}$

$\alpha=26.57^\circ$

$\boxed{\text{Max: }(26.57^\circ, -2+\sqrt{5}) \quad \text{Min: }(206.57^\circ, -2-\sqrt{5})}$ //

4. Given that $\cos\theta=\frac{3}{5}$ and θ is in Q1, then what is the value of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$?

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}} \Rightarrow \sin\frac{\theta}{2}=\sqrt{\frac{1-\frac{3}{5}}{2}}=\sqrt{\frac{\frac{2}{5}}{2}}=\boxed{\frac{\sqrt{5}}{5}} \text{ //}$$

(rationalize $\frac{1}{\sqrt{5}} \to \frac{\sqrt{5}}{5}$)

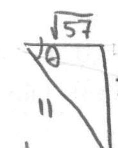
$$\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+\frac{3}{5}}{2}}=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}=\boxed{\frac{2\sqrt{5}}{5}} \text{ //}$$

5. Given that $\sin\theta=\frac{-8}{11}$ and θ is in Q4, then what is the value of $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$?

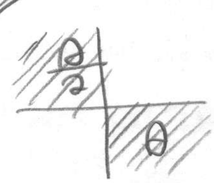
$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}=\sqrt{\frac{1-\frac{\sqrt{57}}{11}}{2}}=\boxed{\sqrt{\frac{11-\sqrt{57}}{22}}} \text{ //} \approx 0.396$$

$\sin\theta=\frac{-8}{11}$

$\cos\theta=\frac{\sqrt{57}}{11}$



$$\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}=\boxed{-\sqrt{\frac{11+\sqrt{57}}{22}}} \text{ //} \approx -0.92$$



When halving an angle in Q4, it ends up in Q2, so cos will be negative & sin will be positive.

6. Prove the following identities:

$\sin^2\frac{1}{2}\theta=\frac{1}{2}(1-\cos\theta)$

$=\left(\pm\sqrt{\frac{1-\cos\theta}{2}}\right)^2$ half angle identity

$=\frac{1-\cos\theta}{2}=\text{RHS}$ //

$\cos^2\frac{1}{2}\theta=\frac{1}{2}(1+\cos\theta)$

$\left(\pm\sqrt{\frac{1+\cos\theta}{2}}\right)^2$ half angle identity

$=\frac{1+\cos\theta}{2}=\text{RHS}$ //

$\cos\theta = \dfrac{1-\tan^2\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$ half angle identity RHS $= \dfrac{1-\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}{1+\left(\frac{1-\cos\theta}{\sin\theta}\right)^2} = \dfrac{\sin^2\theta-(1-\cos\theta)^2}{\sin^2\theta+(1-\cos\theta)^2}$ $= \dfrac{\overset{1-\cos^2\theta}{\cancel{\sin^2\theta}} - 1 + 2\cos\theta - \cos^2\theta}{\underset{1}{\cancel{\sin^2\theta}} + 1 - 2\cos\theta + \cancel{\cos^2\theta}}$ $= \dfrac{2\cos\theta-2\cos^2\theta}{2-2\cos\theta} = \dfrac{\cancel{2}\cos\theta\cancel{(1-\cos\theta)}}{\cancel{2}\cancel{(1-\cos\theta)}} = \cos\theta = \text{LHS}$ // $\boxed{\text{NPV } \cos\frac{1}{2}\theta \neq 0}$	$\cot\theta + \csc\theta = \cot\dfrac{\theta}{2}$ RHS $= \dfrac{1}{\tan\frac{\theta}{2}} = \dfrac{1}{\frac{\sin\theta}{1+\cos\theta}} = \dfrac{1+\cos\theta}{\sin\theta}$ $= \csc\theta + \cot\theta = \text{LHS}$ // $\boxed{\text{NPV: } \sin\theta \neq 0,\ \sin\frac{\theta}{2} \neq 0}$

7. Prove the identity: $\dfrac{\cos x - 5\cos 5x + \cos 9x}{\sin x - 5\sin 5x + \sin 9x} = \cot 5x$

sum to product identity

$$= \frac{2\cos 5x\cos 4x - 5\cos 5x}{2\sin 5x\cos 4x - 5\sin 5x} = \frac{\cos 5x\cancel{(2\cos 4x-5)}}{\sin 5x\cancel{(2\cos 4x-5)}} = \cot 5x = \text{RHS} \ //$$

factor

$\boxed{\begin{array}{l}\text{NPV: } \sin 5x \neq 0 \\ \sin x - 5\sin 5x + \sin 9x \neq 0\end{array}}$

8. Solve for θ, where $0° \le \theta \le 360°$ $\quad \sin 9\theta \sin 8\theta - \sin 7\theta \sin 6\theta = 0$

product to sum $\quad = \frac{1}{2}(\cos 17\theta - \cancel{\cos\theta}) - \frac{1}{2}(\cos 13\theta - \cancel{\cos\theta})$

sum to product $\quad = \frac{1}{2}(\cos 17\theta - \cos 13\theta)$

$= \frac{1}{2} \cdot -2\sin 15\theta \sin 2\theta = 0$

solve $\quad \sin 15\theta \sin 2\theta = 0$

$\sin 15\theta = 0 \qquad \sin 2\theta = 0$

$15\theta = 0, 180, 360, \dots \qquad 2\theta = 0, 180, 360, \dots$

$\boxed{\theta = 0, 12, 24, 36, \dots, 360}$ // $\boxed{\theta = 0, 90, 180, 270, 360}$ //

or → $\theta = 12n;\ 0 \le n \le 30$

9. Solve for θ, where $0° \le \theta \le 360°$ $\cos\theta - \cos 3\theta = \sin 2\theta$

$$\cos a - \cos b = -2\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$$

$$\cos\theta - \cos 3\theta = \sin 2\theta$$

$$-2\sin 2\theta \sin\theta = \sin 2\theta$$

$$\sin 2\theta(-2\sin\theta - 1) = 0$$

$$\sin 2\theta = 0 \qquad \sin\theta = -\tfrac{1}{2}$$

$$\boxed{\theta = 0, 90°, 180°, 270°, 360° \quad \theta_1 = 210°, 330°}$$

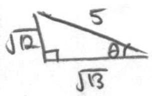
10. If $\dfrac{\sin^2\theta}{1+3\cos^2\theta} = \dfrac{3}{16}$, where $90° < \theta < 180°$, find the value of $\dfrac{\sin\theta}{1+3\cos\theta}$

$$\frac{\sin^2\theta}{1+3(1-\sin^2\theta)} = \frac{3}{16}$$

$$16\sin^2\theta = 3 + 9 - 9\sin^2\theta$$

$$25\sin^2\theta = 12$$

$$\sin^2\theta = \frac{12}{25}$$



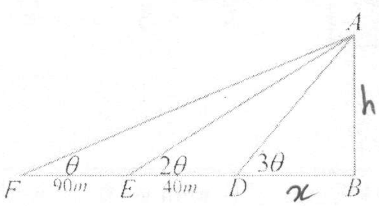
$$\sin\theta = \frac{2\sqrt{3}}{5} \quad (\text{positive b/c } 90° < \theta < 180°)$$

$$\Rightarrow \cos\theta = \frac{-\sqrt{13}}{25}$$

$$\frac{\sin\theta}{1+3\cos\theta} = \frac{\frac{2\sqrt{3}}{5}}{1 - \frac{3\sqrt{13}}{25}} = \frac{\frac{2\sqrt{3}}{5}}{\frac{25-3\sqrt{13}}{25}}$$

$$= \boxed{\frac{10\sqrt{3}}{25-3\sqrt{13}}}$$

11. AB is a building and BDEF is a level straight line. The angles of elevation of the top of the tower from F, E, D, are θ, 2θ, and 3θ respectively. If $EF = 90m$ and $DE = 40m$ find the height of the building. Hint: The following two identities will be helpful: $[\cot\theta - \cot 2\theta = \csc 2\theta]$ and $\left[\cot 2\theta - \cot 3\theta = \frac{1}{2}\sec\theta\csc 3\theta\right]$ (p28 HKCEE).



Step 1

$$\cot\theta - \cot 2\theta = \csc 2\theta$$

$$\frac{130+x}{h} - \frac{40+x}{h} = \csc 2\theta$$

$$\frac{90}{h} = \csc 2\theta$$

$$h = 90\sin 2\theta$$

Step 2

$$\cot 2\theta - \cot 3\theta = \tfrac{1}{2}\sec\theta\csc 3\theta$$

$$\frac{40+x}{h} - \frac{x}{h} = \frac{1}{2\cos\theta\sin 3\theta}$$

$$\frac{h}{40} = 2\cos\theta\sin 3\theta$$

$$h = 80\cos\theta\sin 3\theta$$

$$\frac{h}{h} = \frac{\cancel{90}^{9} \times \cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{80}_{4}\cancel{\cos\theta}\sin 3\theta}$$

$$4\sin 3\theta = 9\sin\theta$$

$$12\sin\theta - 16\sin^3\theta = 9\sin\theta$$

$$16\sin^3\theta - 3\sin\theta = 0$$

$$16\sin^2\theta = 3$$

$$\sin^2\theta = \frac{3}{16}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{4}$$

$$\cos\theta = \pm\frac{\sqrt{13}}{4}$$

$$\Longrightarrow h = 90\sin 2\theta = 180\sin\theta\cos\theta = 180\cdot\frac{\sqrt{13}}{4}\cdot\frac{\sqrt{3}}{4} = \boxed{\frac{45\sqrt{39}}{4}}$$

12. Challenge: Find the general solution of $\tan 5\theta + \cot 2\theta = 0$ (p19 HKCEE)

$$\frac{\sin 5\theta}{\cos 5\theta} + \frac{\cos 2\theta}{\sin 2\theta} = 0$$

$$\frac{\sin 5\theta \sin 2\theta + \cos 2\theta \cos 5\theta}{\cos 5\theta \sin 2\theta} = 0$$

$$\frac{\cos(5\theta - 2\theta)}{\cos 5\theta \sin 2\theta} = 0 \Rightarrow \cos 3\theta = 0$$

$3\theta = 90°, 270°, 450°$

$\theta = 30°, 90°, \ldots$

~~$\theta_1 = 30° + 120k$~~ rej. NPV.

$\boxed{\theta_2 = 90° + 120k}$

13. Find the general solution of $\sin 9x \sin 8x - \sin 7x \sin 6x = 0$ HKCE

(exact same as #8)

same solution provided regradless:

$$= \frac{1}{2}(\cos 17x + \cancel{\cos x}) - \frac{1}{2}(\cos 13x + \cancel{\cos x}) = 0$$

$$\frac{1}{2}(\cos 17x - \cos 13x) = 0$$

$$\frac{1}{2} \cdot -2 \sin 15x \sin 2x = 0$$

$$\sin 15x \sin 2x = 0 \Rightarrow \boxed{\begin{array}{l} x_1 = 12n;\ n \in Z \\ x_2 = 90n;\ n \in Z \end{array}}$$

14. Using the identity: $\cos 3x = \cos(x + 2x)$, prove the following identity: $\cos 3x = 4\cos^3 x - 3\cos x$

$\cos 3x =$	$4\cos^2 x - 3\cos x$
$= \cos(x + 2x)$	
$= \cos x \cos 2x - \sin x \sin 2x$	double angle identity
$= \cos x \cdot (\cos^2 x - \sin^2 x) - 2\sin^2 x \cos x$	double angle identity
$= \cos^3 x - \cos x \sin^2 x - 2\sin^2 x \cos x$	expand
$= \cos^3 x - 3\cos x \sin^2 x$	combine liketerms
$= \cos^3 x - 3\cos x(1 - \cos^2 x)$	pythag. identity
$= \cos^3 x - 3\cos x + 3\cos^3 x$	combine liketerms
$= 4\cos^3 x - 3\cos x$	$=$ RHS //

No NPVs